Int. J. Solids Structures, 1974, Vol. 10, pp. 603–619. Pergamon Press. Printed in Gt. Britain.

VIBRATION OF CONTINUOUS CIRCULAR PLATES

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(*Received* 2 *April* 1973; *revised* 26 *September* 1973)

Abstract—The general development of the theory given here considers the material to be orthotropic and continuous over $(n-1)$ elastic or rigid supports. The effect of rotatory inertia and in-plane loads are also included while formulating the equations of motion. Double and triple series solutions are given for orthotropic continuous plates. By matching the continuity conditions at the intermediate supports and satisfying the boundary conditions at the outer edge, the frequency determinant is obtained. For the purpose of numerical computations, an isotropic plate continuous over an intermediate-rigid or elastic-support and free and with no in-plane loads at the outer edge is considered. It is found that the influence of Poisson's ratio on the frequency parameter is significant only for the first symmetric or asymmetric modes. The rotatory inertia influences the frequency parameter when the radius to thickness ratio is less than 80, viz, when the plate is thick. Moreover, the elasticity of the support influences considerably the free vibration of plates.

NOTATION

a, b_n outer radius of the continuous plate
D $= Eh^3/12(1-\nu^2)$, flexural rigidity of the plate
E Young's modulus of elasticity
I_r, I_θ area moment of inertia in the r- and θ- directions
m number of nodal dia.
N $h\sigma_0$, the radially compressive force per unit width applied at the outer boundary
s number of nodal circles
v $(\Omega-\Omega_s)/\Omega_s x\,100$, Ω_s being the frequency parameter Ω when $\nu = 0{\cdot}3000$
γ mass density per unit volume
ν Poisson's ratio
ρ_s $= b_1/a$, radial distance to the support
σ_0 the applied stress at the outer boundary
ω circular frequency in radians.

INTRODUCTION

Continuous plates are widely used in almost every complicated engineering structure such as aeroframes, marine structures like ships and submarines, pressure vessels, rocket launching pads, instrument mounting bases for space vehicles etc. The influence of a concentric rigid ring support, placed at an arbitrary distance from the centre, on the free vibrations of a thin isotropic circular plate has been studied by Bodine[1, 2]. An allied problem of the vibration of the circular plate stiffened by concentric rings has been investigated by Fleyshman and Shabliy[3] and Kirk and Leissa[4].

The object of the present work is to investigate the free vibrations of isotropic and orthotropic plates continuous over $(n-1)$ intermediate-elastic or rigid-concentric ring supports.

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While formulating the equations of motion the influence of rotatory inertia and in-plane loads are also considered. Double series solutions and triple series solutions are obtained for the orthotropic plates which reduce to known exact solutions in the case of isotropic plates. By matching the continuity conditions and satisfying the boundary conditions, frequency equations are obtained. For the purpose of numerical calculations, an isotropic plate continuous over an intermediate concentric ring support and free at the outer edge is considered. Since in practical cases rigid intermediate supports are only an approximation to the actuality, the study also takes into consideration the influence of the elasticity of support.

GOVERNING EQUATION AND SOLUTION

The equations of motion for the transverse vibration of cylindrically orthotropic circular plate element shown in Fig. 1, with rotatory inertia and in-plane loads are,

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{\theta r}}{\partial \theta} + \frac{N_r - N_\theta}{r} = 0$$

$$\frac{\partial N_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{N_{r\theta} + N_{\theta r}}{r} = 0$$

$$\frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial M_\theta r}{\partial \theta} + \frac{M_r - M_\theta}{r} - Q_r = -\gamma I_r \frac{\partial^3 w}{\partial t^2 \partial r}$$

$$\frac{\partial M_{r\theta}}{\partial r} - \frac{1}{r}\frac{\partial M_\theta}{\partial \theta} + \frac{M_{r\theta} - M_{\theta r}}{r} + Q_\theta = -\frac{\gamma I_\theta}{r}\frac{\partial^3 w}{\partial t^2 \partial \theta}$$

$$\frac{\partial Q_r}{\partial r} + \frac{1}{r}\cdot\frac{\partial Q_\theta}{\partial \theta} + \frac{Q_r}{r} + N_r\frac{\partial^2 w}{\partial r^2} + \frac{N_\theta}{r}\frac{\partial w}{\partial r} + \frac{N_\theta}{r^2}\frac{\partial^2 w}{\partial \theta^2} + 2N_{r\theta}\left(\frac{1}{r^2}\frac{\partial w}{\partial \theta} - \frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial \theta}\right) = \gamma h\frac{\partial^2 w}{\partial t^2} \quad (1)$$

where

$$N_r = h\sigma_r, \qquad N_\theta = h\sigma_\theta, \qquad N_{r\theta} = N_{\theta r} = h\tau_{r\theta}$$

$$M_r = -\frac{S_{11}h^3}{12}\left[\frac{\partial^2 w}{\partial r^2} + a_1\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right]$$

$$M_\theta = -\frac{S_{11}h^3}{12}\left[a_1\frac{\partial^2 w}{\partial r^2} + k^2\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right]$$

$$M_{r\theta} = 2\left(\frac{S_{11}h^3}{12}\right)a_3\left[\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right] = -M_{\theta r}$$

$$Q_r = -\frac{S_{11}h^3}{12}\left[\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{k^2}{r^2}\cdot\frac{\partial w}{\partial r} + \left(\frac{a_1 + 2a_3}{r^2}\right)\frac{\partial^3 w}{\partial r\,\partial \theta^2} - \left(\frac{a_1 + k^2 + 2a_3}{r^3}\right)\frac{\partial^2 w}{\partial \theta^2}\right] + \gamma I_r\frac{\partial^3 w}{\partial t^2\,\partial r} \quad (2)$$

$$Q_\theta = -\frac{S_{11}h^3}{12}\left[\left(\frac{a_1 + 2a_3}{r}\right)\frac{\partial^3 w}{\partial r^2\,\partial \theta} + \frac{k^2}{r^2}\frac{\partial^2 w}{\partial r\,\partial \theta} + \frac{k^2}{r^3}\cdot\frac{\partial^3 w}{\partial \theta^3}\right] - \frac{\gamma I_\theta}{r}\frac{\partial^3 w}{\partial t^2\,\partial \theta}$$

where,

$$a_1 = S_{12}/S_{11}$$
$$k^2 = S_{22}/S_{11} \tag{3}$$
$$a_3 = S_{66}/S_{11}$$

and S_{11}, S_{12}, S_{22}, S_{66} are the elastic stiffness, h is the thickness of the homogeneous plate and γI_r, γI_θ are the mass moment of inertia of the plate in the r- and θ- directions.

Combining equations (1 and 2), the differential equation for the displacement of the middle plane of the plate is obtained as,

$$\frac{S_{11}h^3}{12}\left[\frac{\partial^4 w}{\partial r^4}+\left(\frac{2a_1+4a_3}{r^2}\right)\frac{\partial^4 w}{\partial r^2\,\partial\theta^2}+\frac{k^2}{r^4}\frac{\partial^4 w}{\partial\theta^4}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}\right.$$
$$-\left(\frac{2a_1+4a_3}{r^3}\right)\frac{\partial^3 w}{\partial r\,\partial\theta^2}-\frac{k^2}{r^2}\frac{\partial^2 w}{\partial r^2}+\left(\frac{2a_1+2k^2+4a_3}{r^4}\right)\frac{\partial^2 w}{\partial\theta^2}$$
$$\left.+\frac{k^2}{r^3}\frac{\partial w}{\partial r}\right]+\frac{\partial^2}{\partial t^2}\left[\gamma h w-\frac{\gamma I_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)-\frac{\gamma I_\theta}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right]$$
$$-\left[N_r\frac{\partial^2 w}{\partial r^2}+N_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)+2N_{r\theta}\left(\frac{1}{r^2}\frac{\partial w}{\partial\theta}-\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\theta}\right)\right]=0 \tag{4}$$

where N_r, N_θ and $N_{r\theta}$ are the stress resultants acting on the element of the plate.

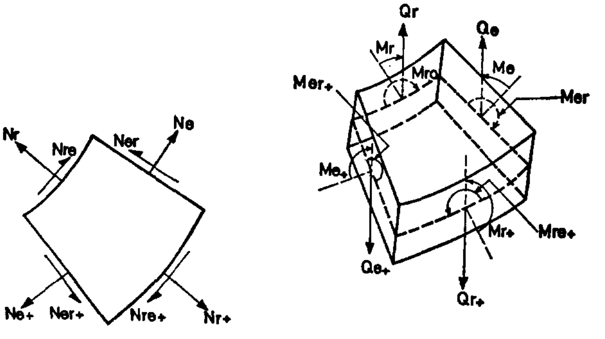

Fig. 1. Forces and moments acting on a plate element.

The sign convention of forces and moments are shown in Fig. 1, along with the coordinate sytems. To generalise the derivations, the above equation is non-dimensionalised by the proper choice of parameters and can be written as,

$$\left[\frac{\partial^4\bar{w}}{\partial\rho^4}+\left(\frac{2a_1+4a_3}{\rho^2}\right)\frac{\partial^4\bar{w}}{\partial\rho^2\,\partial\theta^2}+\frac{k^2}{\rho^4}\frac{\partial^4\bar{w}}{\partial\theta^4}+\frac{2}{\rho}\frac{\partial^3\bar{w}}{\partial\rho^3}\right.$$
$$\left.-\left(\frac{2a_1+4a_3}{\rho^3}\right)\frac{\partial^3\bar{w}}{\partial\rho\,\partial\theta^2}-\frac{k^2}{\rho^2}\frac{\partial^2\bar{w}}{\partial\rho^2}+\left(\frac{2a_1+2k^2+4a_3}{\rho^4}\right)\frac{\partial^2 w}{\partial\theta^2}+\frac{k^2}{\rho^3}\frac{\partial\bar{w}}{\partial\rho}\right]$$
$$+\Omega^2\frac{\partial^2}{\partial\bar{t}^2}\left[\bar{w}-\frac{I\hat{\rho}}{ha^2\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\bar{w}}{\partial\rho}\right)-\frac{I_\theta}{ha^2\rho^2}\frac{\partial^2\bar{w}}{\partial\theta^2}\right.$$
$$-\frac{12b_n^2}{S_{11}h^3}\left[N_\rho\frac{\partial^2\bar{w}}{\partial\rho^2}+N_\theta\left(\frac{1}{\rho}\frac{\partial\bar{w}}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 w}{\partial\theta^2}\right)+2N_{\rho\theta}\left(\frac{1}{\rho^2}\frac{\partial\bar{w}}{\partial\theta}-\frac{1}{\rho}\frac{\partial^2\bar{w}}{\partial\rho\,\partial\theta}\right)\right] \tag{5}$$

where,

$$\bar{w} = w/b_n$$
$$\rho = r/b_n \tag{6}$$
$$\bar{t} = \omega t$$
$$\Omega^2 = \frac{12\gamma h b_n^4 \omega^2}{S_{11}h^3}.$$

The plate is continuous over intermediate ring supports as shown in Fig. 2. The plate continuous over $(n-1)$ intermediate concentric ring supports can be split up into a set of $(n-1)$ annular plates of radii $(b_1, b_2), \ldots (b_{j-1}, b_j), \ldots (b_{n-1}, b_n)$ and a central complete

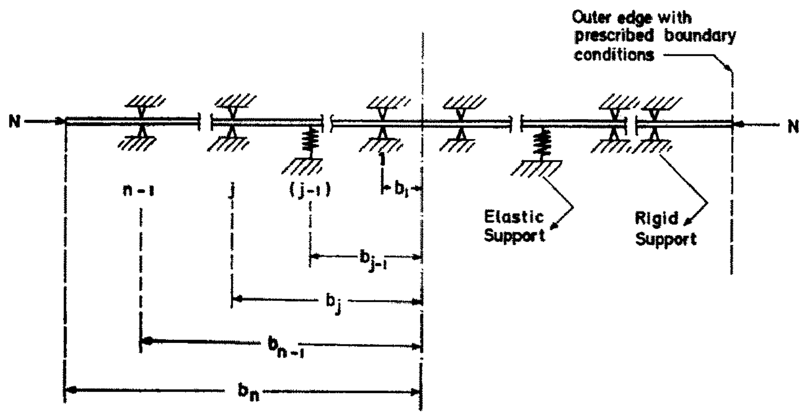

Fig. 2. Continuous plate with $(n-1)$ rigid and elastic ring supports.

plate of radius b_1. The method of approach to obtain the solution for the vibration of continuous plate is based on matching the continuity conditions at each intermediate support and satisfying the boundary conditions at the outer edge. This can be done provided the solutions for the vibration of annular and complete plates with rotatory inertia and in-plane loads are available.

SOLUTION TO ANNULAR PLATE

The direct stresses acting on the jth annular plate and the central complete plate, under the action of radially inward forces at the outer edge, are shown in Fig. 3. The stresses developed in the annular plate due to radially inward and outward stresses of σ_j and σ_{j-1} can be obtained by solving the plane elasticity equations. The expressions for the non-zero stress components are seen to be in the following form:

$$N_\rho = h\sigma_\rho = h[B_1\rho^{k-1} + B_2\rho^{-k-1}]$$
$$N_\theta = h\sigma_\theta = hk[B_1\rho^{k-1} - B_2\rho^{-k-1}] \tag{7}$$
$$N_{\rho\theta} = 0$$

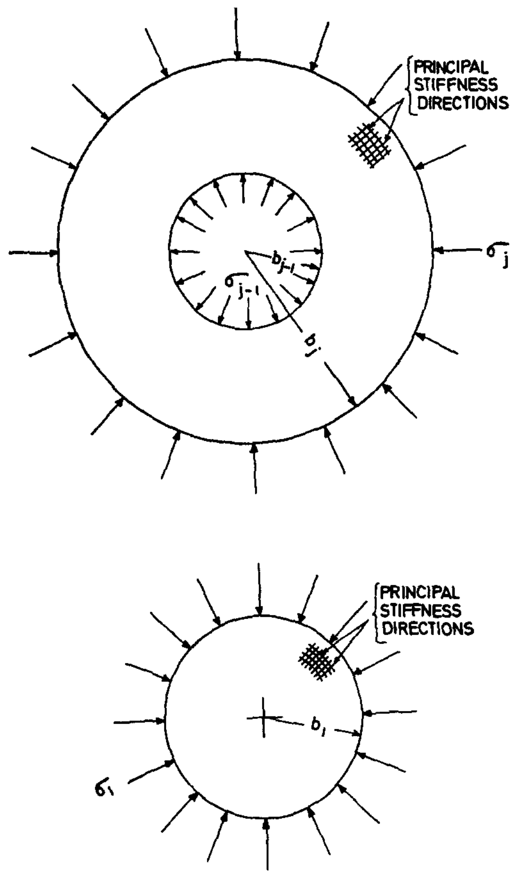

Fig. 3. Annular and complete plates.

where,

$$B_1 = \frac{b_n^{k-1}(\sigma_j b_{j-1}^{-k-1} - \sigma_{j-1} b_j^{-k-1})}{b_{j-1}^{k-1} b_j^{-k-1} - b_{j-1}^{-k-1} b_j^{k-1}}$$

$$B_2 = \frac{b_n^{-k-1}(\sigma_j b_{j-1}^{k-1} - \sigma_{j-1} b_j^{k-1})}{b_{j-1}^{-k-1} b_j^{k-1} - b_{j-1}^{k-1} b_j^{-k-1}} \tag{8}$$

$$\sigma_{j-1} = \sigma_0 \left(\frac{b_{j-1}}{b_n}\right)^{k-1}$$

$$\sigma_j = \sigma_0 \left(\frac{b_j}{b_n}\right)^{k-1}.$$

Assuming a separation of variable type deflection function in the form

$$\bar{w} = R(\rho) \cos m\theta \, e^{it} \tag{9}$$

and substituting equations (7) and (9) into (5), and using the change of variables $\rho = e^u$, we get,

$$\frac{d^4R}{du^4} - 4\frac{d^3R}{du^3} + (5-\lambda^2)\frac{d^2R}{du^2} + (2\lambda^2-2)\frac{dR}{du} + \varepsilon^2 R - \mu_1 e^{(1+k)u}\left[\frac{d^2R}{du^2} + (k-1)\frac{dR}{du} - km^2R\right]$$
$$- \mu_2 e^{(1-k)u}\left[\frac{d^2R}{du^2} - (k+1)\frac{dR}{du} + km^2R\right] + \Omega^2\left[e^{2u}\left(\frac{I_r}{ha^2}\frac{d^2R}{du^2} - \frac{I_\theta}{ha^2}m^2R\right) - e^{4u}R\right] = 0 \tag{10}$$

where,

$$\begin{aligned}\mu_1 &= \frac{12b_n{}^2B_1h}{S_{11}h^3}\\ \mu_2 &= \frac{12b_n{}^2B_2h}{S_{11}h^3}\\ \lambda^2 &= m^2(2a_1+4a_3)+k^2\\ \varepsilon^2 &= m^4k^2 - m^2(2a_1+2k^2+4a_3).\end{aligned} \tag{11}$$

A closed form solution to equation (10) is not seen to be readily available. Equation (10) cannot be solved by the Frobenius method using a single infinite series, since there are three different variables in u, namely, $e^{(1+k)u}$, $e^{(1-k)u}$ and e^{2u} appearing in (10), each giving rise to three completely different series which cannot be incorporated into the single series of Frobenius. Moreover, it is not easily possible to reduce equation (10) to a simple differential equation by any transformations. However, it is seen that a solution can be obtained by modifying the single Frobenius Series into a triple series, each varying independently. Such a series which would lend itself as a solution to (10) can be represented as,

$$R = \sum_{i=0,1}^{\infty}\sum_{j=0,1}^{\infty}\sum_{l=0,1}^{\infty} c_{ijl}e^{[q+i(1+k)+j(1-k)+2l]u} \tag{12}$$

where q represents the unknown roots to be determined. Substituting this into equation (10) and collecting all like powers of the exponent in u and equating them to zero, we get the indicial equation,

$$[q^4 - 4q^3 + (5-\lambda^2)q^2 + (2\lambda^2-2)q + \varepsilon^2]c_{000} = 0 \tag{13}$$

and the recurrence relations,

$$\{[q+(1+k)]^4 - 4[q+(1+k)]^3 + (5-\lambda^2)[q+(1+k)]^2 + (2\lambda^2-2)[q+(1+k)] + \varepsilon^2\}c_{100} - \mu_1\{q^2+(k-1)q-km^2\}c_{000} = 0$$

$$\{[q+(1-k)]^4 - 4[q+(1-k)]^3 + (5-\lambda^2)[q+(1-k)]^2 + (2\lambda^2-2)[q+(1-k)] + \varepsilon^2\}c_{010} - \mu_2\{q^2-(1+k)q+km^2\}c_{000} = 0$$

$$\{(q+2)^4 - 4(q+2)^3 + (5-\lambda^2)(q+2)^2 + (2\lambda^2-2)(q+2) + \varepsilon^2\}c_{001} + \Omega^2\left\{\frac{I_r}{ha^2}q^2 - \frac{I_\theta m^2}{ha^2}\right\}c_{000} = 0$$

$$\{[q + i(1 + k) + j(1 - k) + 2l]^4 - 4[q + i(1 + k) + j(1 - k) + 2l]^3 + (5 - \lambda^2)[q + i(1 + k) + j(1 - k) + 2l]^2 + (2\lambda^2 - 2) \times [q + i(1 + k) + j(1 - k) + 2l] + \varepsilon^2\}c_{ijl} - \mu_1\{[q + (i - 1)(1 + k) + j(1 - k) + 2l]^2 + (k - 1)[q + (i - 1)(1 + k) + j(1 - k) + 2l] - km^2\}c_{i-1,j,l} - \mu_2\{[q + i(1 + k) + (j - 1)(1 - k) + 2l]^2 - (1 + k)[q + i(1 + k) + (j - 1)(1 - k) + 2l] + km^2\}c_{i,j-1,l} + \Omega^2\left\{\frac{I_r}{ha^2}[q + i(1 + k) + j(1 - k) + 2(l - 1)]^2 - \frac{I_\theta m^2}{ha^2}\right\}c_{i,j,l-1} - \Omega^2 c_{i,j,l-2} = 0. \tag{14}$$

The roots of the indicial equation (13) are found to be,

$$\begin{aligned} q_1 &= 1 + \alpha + \beta \\ q_2 &= 1 + \alpha - \beta \\ q_3 &= 1 - \alpha + \beta \\ q_4 &= 1 - \alpha - \beta \end{aligned} \tag{15}$$

where,

$$\begin{aligned} \alpha &= \tfrac{1}{2}[1 + \lambda^2 + 2(\lambda^2 + \varepsilon^2)^{1/2}]^{1/2} \\ \beta &= \tfrac{1}{2}[1 + \lambda^2 - 2(\lambda^2 + \varepsilon^2)^{1/2}]^{1/2} \end{aligned} \tag{16}$$

Hence the solution to equation (5) can be written as,

$$\bar{w} = [A_1\phi_1(\mu_1, \mu_2, \Omega, \rho) + A_2\phi_2(\mu_1, \mu_2, \Omega, \rho) + A_3\phi_3(\mu_1, \mu_2, \Omega, \rho) + A_4\phi_4(\mu_1, \mu_2, \Omega, \rho)]\cos m\theta\, e^{it} \tag{17}$$

where,

$$\begin{aligned} \phi_1(\mu_1, \mu_2, \Omega, \rho) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_{ijl}\,\rho^{[q_1 + i(1+k) + j(1-k) + 2l]} \\ \phi_2(\mu_1, \mu_2, \Omega, \rho) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_{ijl}\,\rho^{[q_2 + i(1+k) + j(1-k) + 2l]} \\ \phi_3(\mu_1, \mu_2, \Omega, \rho) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_{ijl}\,\rho^{[q_4 + i(1+k) + j(1-k) + 2l]} \\ \phi_4(\mu_1, \mu_2, \Omega, \rho) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_{ijl}\,\rho^{[q_3 + i(1+k) + j(1-k) + 2l]} \end{aligned} \tag{18}$$

The coefficients c_{ijl} are obtained by using the recurrence relations (14). Similar solutions can be obtained for the other $(n - 2)$ annular portions of the continuous plate.

SOLUTION TO COMPLETE PLATE

The solution to the complete plate shown in Fig. 3 is obtained as discussed below. The stresses σ_ρ and σ_θ are obtained by solving the plane elasticity equations of a complete plate subjected to radially inward loads. The solutions are,

$$\begin{aligned} N_\rho &= h\sigma_\rho = -h\sigma_0\rho^{k-1} \\ N_\theta &= h\sigma_\theta = -kh\sigma_0\rho^{k-1}. \end{aligned} \tag{19}$$

Assuming a deflection function of the form given by equation (9) and again making the change of variable $\rho = e^u$, we get,

$$\frac{d^4R}{du^4} - 4\frac{d^3R}{du^3} + (5-\lambda^2)\frac{d^2R}{du^2} + (2\lambda^2-2)\frac{dR}{du} + \varepsilon^2 R$$
$$+ \mu_3 e^{(1+k)u}\left[\frac{d^2R}{du^2} + (k-1)\frac{dR}{du} - km^2R\right]$$
$$+ \Omega^2\left[e^{2u}\left(\frac{I_r}{ha^2}\frac{d^2R}{du^2} - \frac{I_\theta}{ha^2}m^2R\right) - e^{4u}R\right] = 0 \tag{20}$$

where,

$$\mu_3 = \frac{12b_n^2 h\sigma_0}{S_{11}h^3}\,. \tag{21}$$

Proceeding in the same way as for equation (12), equation (20) is solved by assuming a double series of the form,

$$R = \sum_{i=0,\,1}^{\infty}\sum_{j=0,\,1}^{\infty} d_{ij}\, e^{[q+(1+k)i+2j]u}. \tag{22}$$

Substituting the above into equation (20) and collecting the similar exponents in u and equating them to zero, we get the same indicial equation (13), as previously obtained for the annular portion.

The recurrence relations are given as,

$$\{[q+(1+k)]^4 - 4[q+(1+k)]^3 + (5-\lambda^2)[q+(1+k)]^2$$
$$+ (2\lambda^2-2)[q+(1+k)] + \varepsilon^2\}d_{10} + \mu_3[q^2+(k-1)q-km^2]\,d_{00} = 0$$
$$\{(q+2)^4 - 4(q+2)^3 + (5-\lambda^2)(q+2)^2 + (2\lambda^2-2)(q+2) + \varepsilon^2\}d_{01}$$
$$+ \Omega^2\left[\frac{I_r}{ha^2}q^2 - \frac{I_\theta}{ha^2}m^2\right]d_{00} = 0$$
$$\{[q+i(1+k)+2j]^4 - 4[q+i(1+k)+2j]^3 + (5-\lambda^2)[q+i(1+k)+2j]^2$$
$$+ (2\lambda^2-2)[q+i(1+k)+2j] + \varepsilon^2\}\,d_{ij} + \mu_3\{[q+(i-1)(1+k)+2j]^2$$
$$+ (k-1)[q+(i-1)(1+k)+2j] - km^2\}d_{i-1,\,j}$$
$$+ \Omega^2\left\{\frac{I_r}{ha^2}[q+i(1+k)+2(j-1)]^2 - \frac{I_\theta}{ha^2}m^2\right\}d_{i,\,j-1} - \Omega^2\,d_{i,j-2} = 0. \tag{23}$$

Hence the solution to equation (5) can be written down as,

$$\bar{w} = [A_1\psi_1(\mu_3,\Omega,\rho) + A_2\psi_2(\mu_3,\Omega,\rho) + A_3\psi_3(\mu_3,\Omega,\rho) + A_4\psi_4(\mu_3,\Omega,\rho)]\cos m\theta\, e^{it} \tag{24}$$

where,

$$\psi_1(\mu_3,\Omega,\rho) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_{ij}\,\rho^{[q_1+i(1+k)+2j]}$$
$$\psi_2(\mu_3,\Omega,\rho) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d'_{ij}\,\rho^{[q_2+i(1+k)+2j]}$$
$$\psi_3(\mu_3,\Omega,\rho) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d''_{ij}\,\rho^{[q_3+i(1+k)+2j]}$$
$$\psi_3(\mu_3,\Omega,\rho) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d'''_{ij}\,\rho^{[q_4+i(1+k)+2j]} \tag{25}$$

The coefficients d_{ij}s are obtained by using the recurrence relations (23). For a complete plate, considering the finiteness of the deflection function throughout the plate including the center, the deflection configuration given by (24) shall be modified as,

$$\bar{w} = [A_1\psi_1(\mu_3, \Omega, \rho) + A_2\psi_2(\mu_3, \Omega, \rho)] \cos m\theta\, e^{it}. \quad (26)$$

REDUCTION TO ISOTROPIC PLATES

For isotropic plates, the series solutions given in equations (17) and (26) reduce easily to exact solutions in terms of Bessel functions. For isotropic material, the physical parameters will get reduced to,

$$\begin{aligned} k^2 &= 1, \qquad a_1 = \nu, \qquad a_3 = \frac{1-\nu}{2} \\ I_r &= \frac{h^3}{12}, \qquad I_\theta = \frac{h^3}{12}, \qquad S_{11} = \frac{E}{(1-\nu^2)} \end{aligned} \quad (27)$$

and the roots of the indicial equation become,

$$q_1 = m+2, \qquad q_2 = m, \qquad q_3 = -m+2, \qquad q_4 = -m. \quad (28)$$

Substituting the above parameters and roots into equation (17), we obtain in the case of an annular isotropic plate,

$$\bar{w} = [AJ_m(\delta\rho) + BY_m(\delta\rho) + CI_m(\eta\rho) + DK_m(\eta\rho)] \cos m\theta\, e^{i\omega t} \quad (29)$$

where,

$$\begin{aligned} \delta &= \left\{\left[\left(\frac{\mu}{2} + \frac{h^2\Omega^2}{24a^2}\right)^2 + \Omega^2\right]^{1/2} + \left(\frac{\mu}{2} + \frac{h^2\Omega^2}{24a^2}\right)\right\}^{1/2} \\ \eta &= \left\{\left[\left(\frac{\mu}{2} + \frac{h^2\Omega^2}{24a^2}\right)^2 + \Omega^2\right]^{1/2} - \left(\frac{\mu}{2} + \frac{h^2\Omega^2}{24a^2}\right)\right\}^{1/2} \\ \mu &= \frac{12a^2(1-\nu^2)h\sigma_0}{Eh^3}. \end{aligned} \quad (30)$$

For a complete isotropic plate, the solution obtained from equation (26) is

$$\bar{w} = [AJ_m(\delta\rho) + CI_m(\eta\rho)] \cos m\theta\, e^{it}. \quad (31)$$

BOUNDARY AND CONTINUITY CONDITIONS

At $\rho = 1$,

Plate fixed at the boundary,

$$\bar{w} = 0$$
$$\frac{\partial \bar{w}}{\partial \rho} = 0. \quad (32a)$$

Plate simply supported at the boundary,

$$\bar{w} = 0$$
$$M\rho = 0. \quad (32b)$$

Plate free at the boundary,

$$M_\rho = 0$$
$$V_\rho = Q_\rho - \frac{\partial M_{\rho\theta}}{\rho\partial\theta} = 0. \tag{32c}$$

The continuity conditions can be specified in the following manner. Consider the plate as supported over $(n-1)$ intermediate supports. The intermediate supports could be considered as rigid or elastic. At any of the intermediate support $\rho = \rho_j$,

For a rigid support,

$$\begin{aligned}(\bar{w})_{j-} &= (\bar{w})_{j+} = 0\\ \left(\frac{\partial \bar{w}}{\partial \rho}\right)_{j-} &= \left(\frac{\partial \bar{w}}{\partial \rho}\right)_{j+}\\ (M_\rho)_{j-} &= (M_\rho)_{j+}.\end{aligned} \tag{33}$$

For an elastic support,

$$\begin{aligned}(\bar{w})_{j-} &= (\bar{w})_{j+}\\ \left(\frac{\partial w}{\partial \rho}\right)_{j-} &= \left(\frac{\partial \bar{w}}{\partial \rho}\right)_{j+}\\ (M_\rho)_{j-} &= (M_\rho)_{j+}\\ (V_\rho)_{j+} - (V_\rho)_{j-} &= k_1(\bar{w})_{j-}\end{aligned} \tag{34}$$

Using the boundary conditions of (32 a, b or c) and the continuity conditions of (33) or (34) in equations (17) and (26) or (29) and (31), we get $(4n-2)$ equations containing $(4n-2)$ arbitrary constants. These $(4n-2)$ equations lead to the frequency equation which must be solved to find the eigenvalues of the continuous plate. From the eigenvalues, the eigenvectors and the non-dimensionalised deflection modes can be calculated.

EXAMPLE PROBLEMS AND FREQUENCY EQUATIONS

For the purpose of numerical calculations, two specific problems shown in Fig. 9 are considered. Plates are assumed to be isotropic in the calculations. The frequency equations for both the cases are given as equations (35) and (37) [$\mu = 0$].

For the rigidly supported free continuous plate, the frequency equation is,

$$\begin{vmatrix} J''_m(\delta\rho_s) & I''_m(\eta\rho_s) & -J''_m(\delta\rho_s) & -Y''_m(\delta\rho_s) & -I''_m(\eta\rho_s) & -K''_m(\eta\rho_s)\\ J'_m(\delta\rho_s) & I'_m(\eta\rho_s) & -J'_m(\delta\rho_s) & -Y'_m(\delta\rho_s) & -I'_m(\eta\rho_s) & -K'_m(\eta\rho_s)\\ J_m(\delta\rho_s) & I_m(\eta\rho_s) & 0 & 0 & 0 & 0\\ 0 & 0 & J_m(\delta\rho_s) & Y_m(\delta\rho_s) & I_m(\eta\rho_s) & K_m(\eta\rho_s)\\ 0 & 0 & J''_m(\delta\rho_0) & Y''_m(\delta\rho_0) & I''_m(\eta\rho_0) & K''_m(\eta\rho_0)\\ 0 & 0 & J'''_m(\delta\rho_0) & Y'''_m(\delta\rho_0) & I'''_m(\eta\rho_0) & K'''_m(\eta\rho_0) \end{vmatrix} = 0. \tag{35}$$

where

$$\rho_0 = 1$$
$$J'_m(\delta\rho) = \frac{mJ_m(\delta\rho)}{\rho} - \delta J_{m+1}(\delta\rho)$$

$$Y'_m(\delta\rho) = \frac{mY_m(\delta\rho)}{\rho} - \delta Y_{m+1}(\delta\rho)$$

$$I'_m(\eta\rho) = \frac{mI_m(\eta\rho)}{\rho} + \eta I_{m+1}(\eta\rho)$$

$$K'_m(\eta\rho) = \frac{mK_m(\eta\rho)}{\rho} - \eta K_{m+1}(\eta\rho)$$

$$J''_m(\delta\rho) = \left[\frac{(1-\nu)(m^2-m)}{\rho^2} - \delta^2\right] J_m(\delta\rho) + \frac{(1-\nu)\delta}{\rho} J_{m+1}(\delta\rho)$$

$$Y''_m(\delta\rho) = \left[\frac{(1-\nu)(m^2-m)}{\rho^2} - \delta^2\right] Y_m(\delta\rho) + \frac{(1-\nu)\delta}{\rho} Y_{m+1}(\delta\rho)$$

$$I''_m(\eta\rho) = \left[\frac{(1-\nu)(m^2-m)}{\rho^2} + \eta^2\right] I_m(\eta\rho) - \frac{(1-\nu)\eta}{\rho} I_{m+1}(\eta\rho)$$

$$K''_m(\eta\rho) = \left[\frac{(1-\nu)(m^2-m)}{2} + \eta^2\right] K_m(\eta\rho) + \frac{(1-\nu)\eta}{\rho} K_{m+1}(\eta\rho)$$

$$J'''_m(\delta\rho) = \left[-\frac{(1-\nu)(m^3-m^2)}{\rho^3} - \frac{\delta^2 m}{\rho}\right] J_m(\delta\rho) + \left[\frac{(1-\nu)\delta m^2}{\rho^2} + \delta^3\right] J_{m+1}(\delta\rho)$$

$$Y'''_m(\delta\rho) = \left[-\frac{(1-\nu)(m^3-m^2)}{\rho^3} - \frac{\delta^2 m}{\rho}\right] Y_m(\delta\rho) + \left[\frac{(1-\nu)\delta m^2}{\rho^2} + \delta^3\right] Y_{m+1}(\delta\rho)$$

$$I'''_m(\eta\rho) = \left[-\frac{(1-\nu)(m^3-m^2)}{\rho^3} + \frac{\eta^2 m}{\rho}\right] I_m(\eta\rho) + \left[-\frac{(1-\nu)\eta m^2}{\rho^2} + \eta^3\right] I_{m+1}(\eta\rho)$$

$$K'''_m(\eta\rho) = \left[-\frac{(1-\nu)(m^3-m^2)}{\rho^3} + \frac{\eta^2 m}{\rho}\right] K_m(\eta\rho) + \left[\frac{(1-\nu)\eta m^2}{\rho^2} - \eta^3\right] K_{m+1}(\eta\rho) \quad (36)$$

For an elastically supported, free plate, the frequency equation is,

$$\begin{vmatrix} J''_m(\delta\rho_s) & I''_m(\eta\rho_s) & -J''_m(\delta\rho_s) & -Y''_m(\delta\rho_s) & -I''_m(\eta\rho_s) & -K''_m(\eta\rho_s) \\ J'_m(\delta\rho_s) & I'_m(\eta\rho_s) & -J'_m(\delta\rho_s) & -Y'_m(\delta\rho_s) & -I'_m(\eta\rho_s) & -K'_m(\eta\rho_s) \\ J_m(\delta\rho_s) & I_m(\eta\rho_s) & -J_m(\delta\rho_s) & -Y_m(\delta\rho_s) & -I_m(\eta\rho_s) & -K_m(\eta\rho_s) \\ {}^aJ_m(\delta\rho_s) & {}^aI_m(\eta\rho_s) & {}^bJ_m(\delta\rho_s) & {}^bY_m(\delta\rho_s) & {}^bI_m(\eta\rho_s) & {}^bK_m(\eta\rho_s) \\ 0 & 0 & J''_m(\delta\rho_0) & Y''_m(\delta\rho_0) & I''_m(\eta\rho_0) & K''_m(\eta\rho_0) \\ 0 & 0 & J'''_m(\delta\rho_0) & Y'''_m(\delta\rho_0) & I'''_m(\eta\rho_0) & K'''_m(\eta\rho_0) \end{vmatrix} = 0 \quad (37)$$

where,

$${}^aJ'''_m(\delta\rho) = -\left[\frac{m\delta^2}{k_1\rho} - 1\right] J_m(\delta\rho) - \frac{\delta^3}{k_1} J_{m+1}(\delta\rho)$$

$${}^aI'''_m(\eta\rho) = -\left[+\frac{m\eta^2}{k_1\rho} - 1\right] Y_m(\eta\rho) - \frac{\eta^3}{k_1} Y_{m+1}(\eta\rho)$$

$${}^aJ'''_m(\delta\rho) = -\frac{m\delta^2}{k_1\rho} J_m(\delta\rho) + \frac{\delta^3}{k_1} J_{m+1}(\delta\rho),$$

$${}^bY'''_m(\delta\rho) = -\frac{m\delta^2}{k_1\rho} Y_m(\delta\rho) + \frac{\delta^3}{k_1} J_{m+1}(\delta\rho)$$

$$
\begin{aligned}
{}^{b}I_m'''(\eta\rho) &= +\frac{m\eta^2}{k_1\rho} I_m(\eta\rho) + \frac{\eta^3}{k_1} I_{m+1}(\eta\rho) \\
{}^{b}K_m'''(\eta\rho) &= +\frac{m\eta^2}{k_1\rho} K_m(\eta\rho) - \frac{\eta^3}{k_1} K_{m+1}(\eta\rho).
\end{aligned}
\tag{38}
$$

NUMERICAL RESULTS AND DISCUSSION

Numerical calculations were done in an IBM 7044/1401 computer. The program was written in Fortran Language and three subroutines—one each for evaluating the elements of the determinant, the Bessel functions and the determinant value with or without eigenvectors—were used along with the main program. The accuracy imposed on the calculation of Bessel functions was 10^{-8}. The iteration process for calculating each eigenvalue and the corresponding deflection mode took 8 sec on the average.

Examining the continuity conditions in equations (33) and (34), it can be seen that the physical conditions of bending moment and shear can be replaced by the second and third derivatives with respect to ρ of the deflection function $\bar{w}$. From equation (35) it can be seen that the physical parameters that influence the frequency parameter Ω are ν, the Poisson's ratio, h/a, the thickness ratio and ρ_s, the distance of the concentric ring support from the centre. In the case of a free plate on elastic support, the parameters are ν, h/a, ρ_s and k_1, the coefficient of support elasticity.

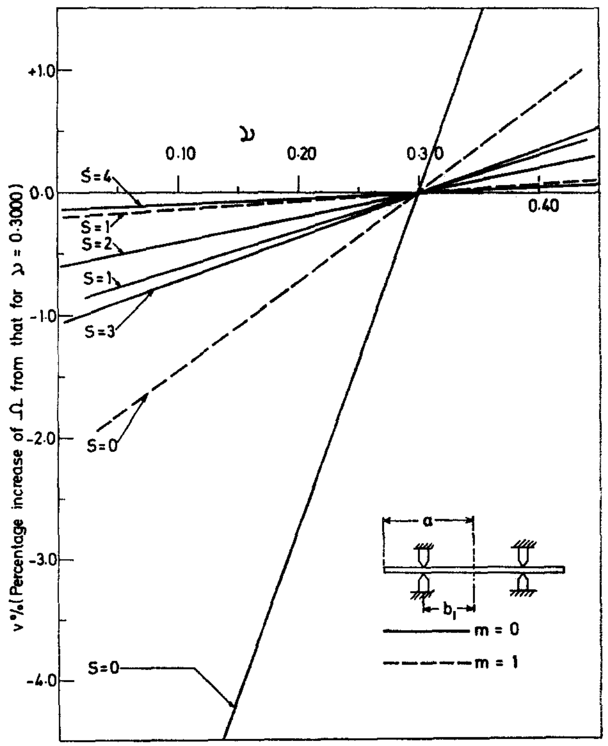

Fig. 4. Influence of Poisson's ratio "ν" on the frequency parameter "Ω" ($\rho_s = 0{\cdot}60$).

Figure 4 shows the influence of Poisson's ratio on the frequency parameter of a free continuous plate on rigid support. The percentage difference in the parameter to the standard value of Ω when $\nu = 0{\cdot}30$ is plotted on the y-axis against ν on the x-axis. As seen the variation is linear in the case of symmetric and first asymmetric modes ($m = 0$ and $m = 1$). For higher modes it is parabolic as could be proved from equation (35). Ω increases with an increase of ν. The first eigenvalue of both the modes (i.e. $s = 0$, $m = 0$ and $s = 0$, $m = 1$) are found to give greater differences than the subsequent eigenvalues. The difference is found to be $\pm 1{\cdot}36$ per cent for a difference of $\pm 0{\cdot}05$ in ν for the lowest mode ($m = 0$, $s = 0$). For the higher frequencies, the difference is small and thus can be neglected.

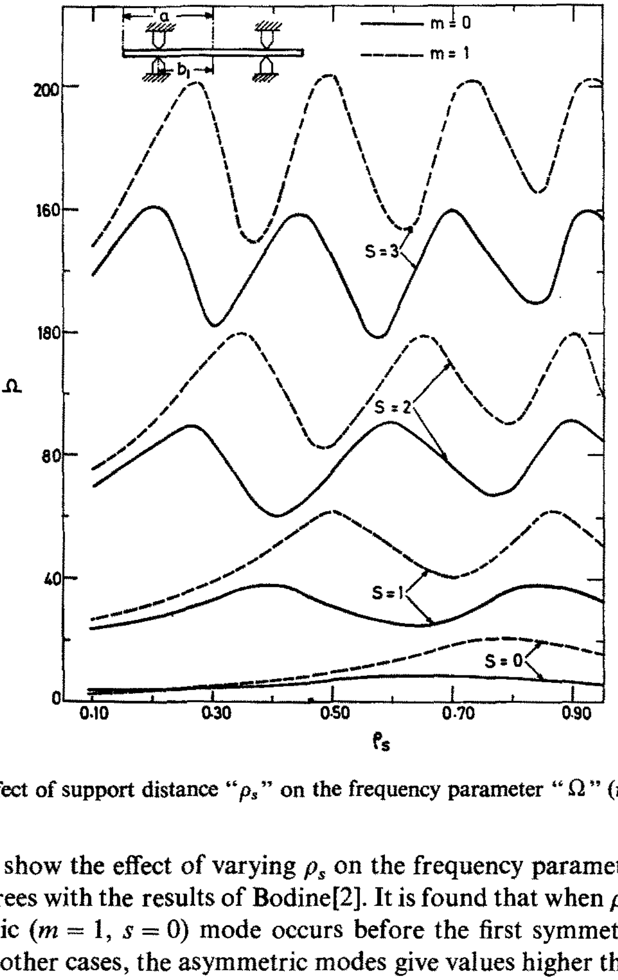

Fig. 5. Effect of support distance "ρ_s" on the frequency parameter "Ω" ($\nu = 0{\cdot}30$).

Figures 5 and 6 show the effect of varying ρ_s on the frequency parameter Ω. The results shown in Fig. 5 agrees with the results of Bodine[2]. It is found that when ρ_s is less than 0·25, the first asymmetric ($m = 1$, $s = 0$) mode occurs before the first symmetric mode ($m = 0$, $s = 0$). For all the other cases, the asymmetric modes give values higher than the symmetric mode. Figure 6 shows the results obtained for the case of an elastically supported plate ($\nu = 0{\cdot}300$, $k_1 = 170{\cdot}0$). Comparing the results in this figure with those given in 5, it can be said that the support elasticity influences the frequency spectrum considerably. The well-defined crests and troughs seen for the rigidly supported plates are not present here, except in the case of the first and second symmetric and asymmetric modes ($m = 0$, $s = 0$; $m = 0$, $s = 1$; $m = 1$, $s = 0$; $m = 1$, $s = 1$). For the higher modes the elasticity of the support does

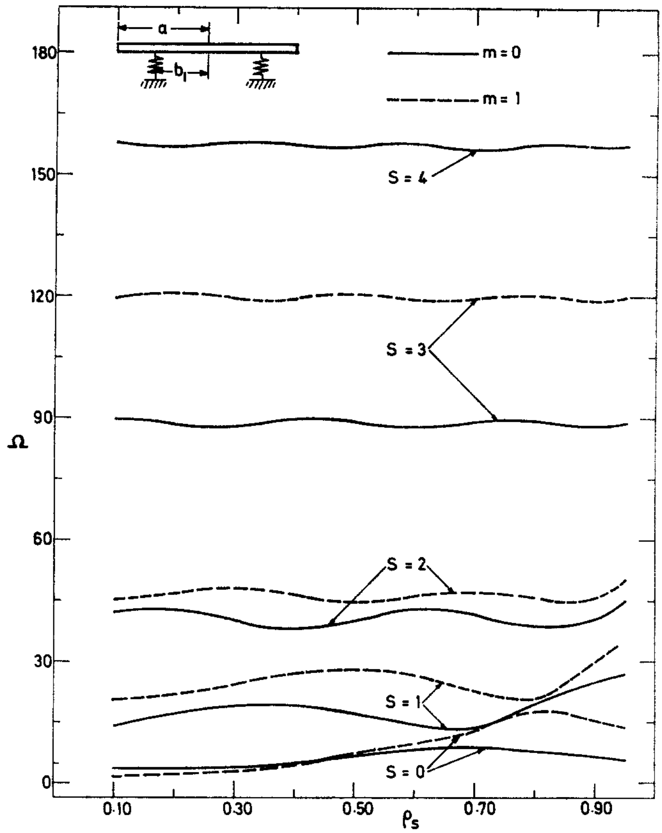

Fig. 6. Influence of support distance "ρ_s" on the frequency parameter "Ω" ($\nu = 0{\cdot}30$, $k_1 = 170$).

not give rise to much differences in Ω when the support moves outward from the centre. It is quite important to note that for ρ_s less than 0·46, the first asymmetric mode ($m = 1$, $s = 0$) gives lower values than the first symmetric mode ($m = 0$, $s = 0$).

Figures 7 and 8 show the frequency spectrums for the two support conditions of interest. From these figures the influence of rotatory inertia on frequency parameters can be estimated. At low values of Ω for any a/h ratio the rotatory inertia effect is seen to be very small. But for higher modes of vibration, the effect is seen to be discernible when a/h is smaller than 100. If the effect of rotatory inertia is not taken into account in the study, then all the curves in Figs. 7 and 8 must be horizontal. For the twelvth symmetric mode ($s = 11$, $m = 0$) of the elastically supported plate with $\frac{a}{h} = 20$, the influence of rotatory inertia is about 22 per cent.

Figure 9 gives some of the interesting vibration modes for the rigidly and elastically supported plates. For the rigidly supported plate, the support is always a circle of zero deflection. Unlike in the case of a free plate[5], the frequency parameter continuously increases with the number of nodal diameters and circles. As could be seen, the elasticity of the support affects the mode shapes considerably. There is always an upward or downward deflection of the elastic support. The second symmetric mode of the elastically supported plate

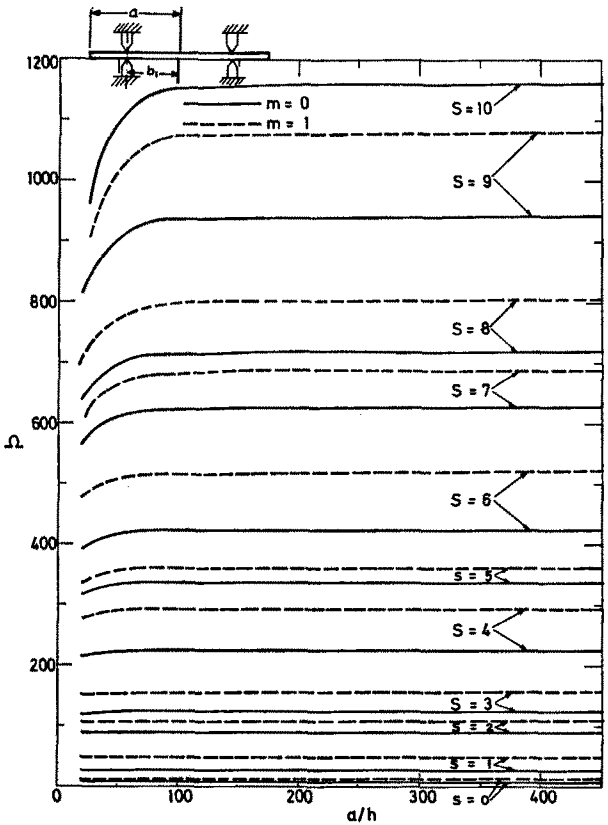

Fig. 7. Effect of rotatory inertia (a/h) on the frequency parameter "Ω" $(\rho_s = 0{\cdot}60,\ \nu = 0{\cdot}3)$.

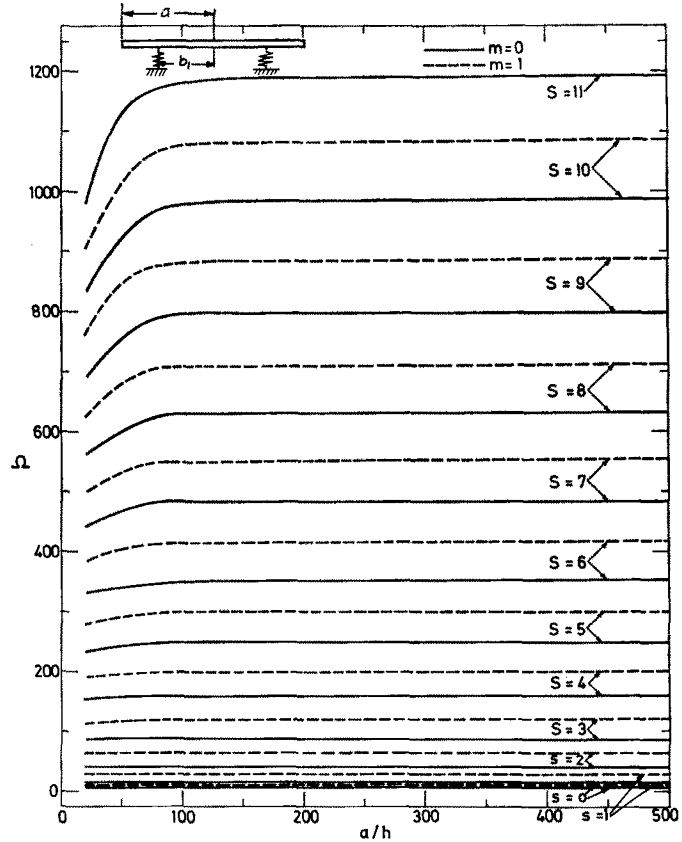

Fig. 8. Influence of rotatory inertia (a/h) on the frequency parameter "Ω" $(\rho_s = 0{\cdot}60,\ \nu = 0{\cdot}30,\ k_1 = 170)$.

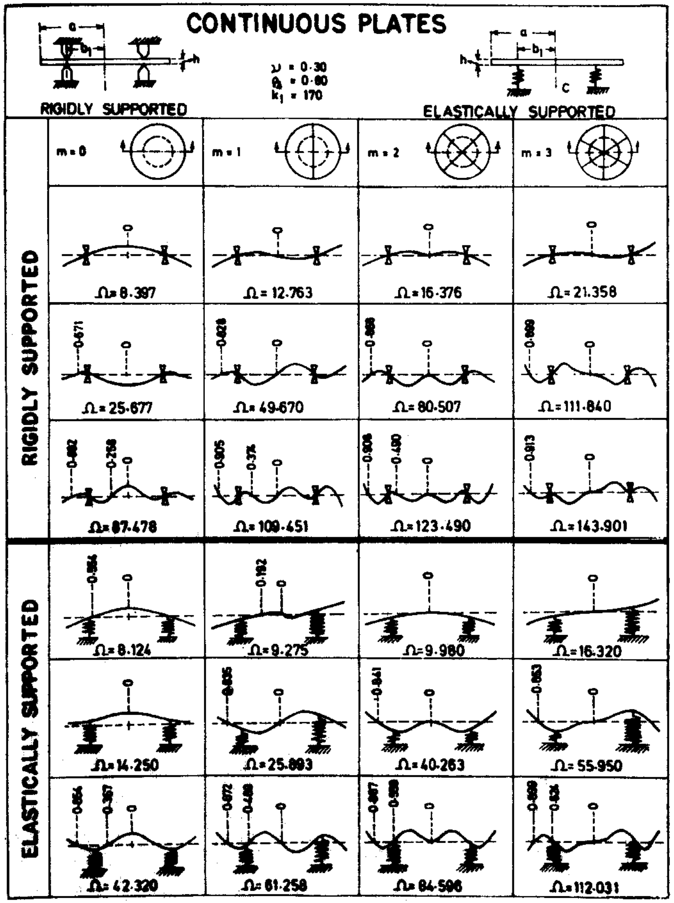

Fig. 9. Normal modes for rigidly and elastically supported plates.

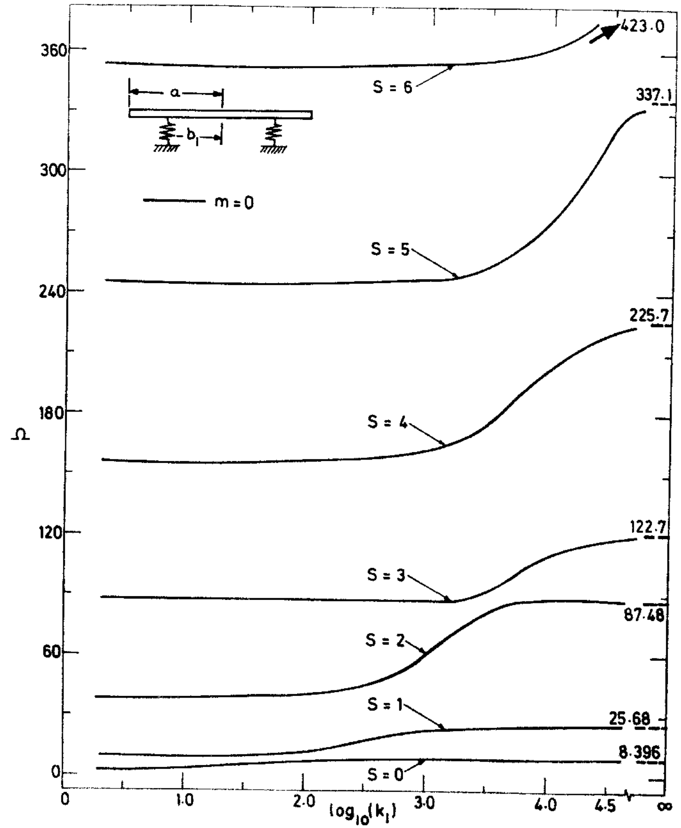

Fig. 10. Effect of stiffness of the elastic support on the frequency parameter "Ω" ($\rho_s = 0{\cdot}60$, $\nu = 0{\cdot}30$).

does not have any nodal circle. The first asymmetric mode ($m = 1$) has a nodal circle within the elastic support while the second asymmetric mode ($m = 1$) again has only one nodal circle which is now located outside the elastic support.

Figure 10 gives the influence of the elasticity of the support on the frequency parameter. It is found that for the lower modes, a value of 10,000 for k_1 (i.e. $\log_{10} k_1 = 4$), it is found to behave almost like a fixed support; and at higher modes even this value of k_1 affects the results very much. For higher modes it is also found that Ω is almost stationary up to $k_1 = 1000$ (i.e. $\log_{10} k_1 = 3$) and then increases rapidly for higher values,

It may be mentioned that the method developed in this work can be applied to stepped circular plates as well.

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Абстракт—По теории приведенной в этой работе материал считается ортотропическим и неразрывным над ($n - 1$) эластичной или жесткой опорами. Для формуляции уравнений движения также включаются эффекты вращательной инерции и эффекты нагрузки по плоскости. Для ортотропических неразрывных пластин нашли двухрядные и трехрядные решения. Согласованием условий неразрывности у промежуточных опор и удовлетворением граничных условий на наружном краю, получили определитель частоты. Для числового расчета рассматривается изотропическая неразрывная пластина над промежуточной эластичной или жесткой опорами, без плоскостных нагрузок на наружный край. Нашли, что влияние коэффициента Пуассона на параметры частоты является существенным только для первых симметричных или асимметричных колебаний. Вращательная инерция влияет на параметр частоты тогда, когда радиус по отношению толщины менее 80, а именно, если лист толстый. Кроме того, эластичность опоры заметно влияет на свободное колебание пластин.